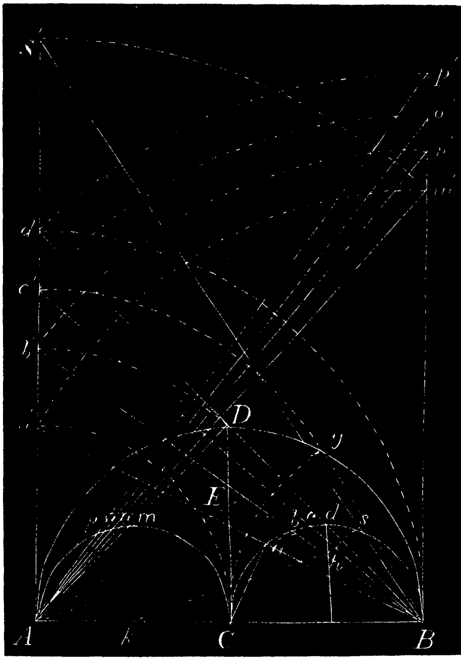
Solution.—Let AB be the side of any given square, upon which describe a semicircle, and on each radius AC and CB, describe other semicircles as shown in the diagram.



From B as centre and radius BC describe the arc Cg. Join Ag cutting the radius CD in E. Then, assuming $AB=1$, we have by similar triangles $Ag^2 : AB^2 :: AC^2 : AE^2$ or $1 - \frac{1}{4} : 1 :: \frac{1}{4} : \frac{1}{3}$ Hence the square described on $AE = \frac{1}{3}$ the square on AB.

Again, from A as centre and radius AC describe the arc Ca'—Aa' being perpendicular to AB. Join Ba' cutting the semicircle on CB in a. Then because $Ba'^2 : AB^2 :: CB^2 : Ba^2$, therefore $1 + \frac{1}{4} : 1 :: \frac{1}{4} : \frac{1}{5}$; or the square desribed on $Ba = \frac{1}{5}$ the square on AB.

In a similar manner, $Bb = \frac{1}{6}$, $Bc = \frac{1}{7}$, $Bd = \frac{1}{8}$, $Am = \frac{1}{9}$, $An = \frac{1}{10}$, $Ao = \frac{1}{11}$, $Ap = \frac{1}{12}$ &c., of the square on AB; and there is a regular law governing the construction as shown in the diagram.

Again, suppose we wanted to construct a square $= \frac{7}{13}$ of a given square. From the above construction we know that $Bs^2 = \frac{1}{13}$ of AB^2 and $Bc^2 = \frac{1}{7}$ of AB^2. With radius Bs describe the arc sh cutting Bc in h. Join Ac and draw through h a line parallel with Ac cutting AB in k; then $Bk^2 = \frac{7}{13}$ of AB^2. Because by similar triangles $Bc^2 : BA^2 :: Bh^2 = Bs^2 : Bk^2$, or, multiplying the antecedents by 7, $7Bc^2 : BA^2 :: 7Bh^2 : Bk^2$. But by construction $7Bc^2 = BA^2$; $\therefore 7Bh^2 = Bk^2$, and because $Bh^2 = \frac{1}{13}$ of $BA^2 \therefore Bk^2 = \frac{7}{13}$ of BA^2, and similarly for any other fractional part.

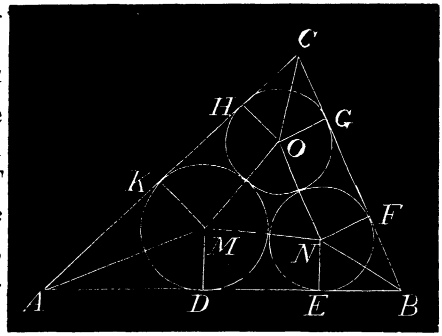
It is evident that a circle, or any regular figure, may be divided in similar fractional parts by the same construction.

SOLUTION OF A PROBLEM.

BY E. B. SEITZ, GREENVILLE, OHIO.

To determine the radii of three circles inscribed in a triangle whose sides are a, b, c, each cir. touching the other two, and also two sides of the triangle.

Solution.—Let ABC be the triangle, M, N, O, the centers of the circles, D, E, F, G, H, K, the points of tangency.



Put $MD = x$, $NE = r_1^2 x$, $OG = r_2^2 x$, and let $r =$ the radius of the inscribed circle of the triangle. Then $AD = AK = x \cot \frac{1}{2}A$, $BE = BF = r_1^2 x \cot \frac{1}{2}B$, $CG = CH = r_2^2 x \times \cot \frac{1}{2}C$, $DE = 2r_1 x$, $HK = 2r_2 x$, $FG = 2r_1 r_2 x$, and we obtain the following equations.

$$x \cot \tfrac{1}{2}A + 2r_1 x + r_1^2 x \cot \tfrac{1}{2}B = c, \ldots\ldots\ldots \quad (1)$$

$$x \cot \tfrac{1}{2}A + 2r_2 x + r_2^2 x \cot \tfrac{1}{2}C = b, \ldots\ldots\ldots \quad (2)$$

$$r_1^2 x \cot \tfrac{1}{2}B + 2r_1 r_2 x + r_2^2 x \cot \tfrac{1}{2}C = a. \ldots\ldots\ldots \quad (3)$$

By Trigonometry we have $b(\cot \frac{1}{2}A - \tan \frac{1}{2}B) = c(\cot \frac{1}{2}A - \tan \frac{1}{2}C)$, . (4)

$$ar_1^2(\cot \tfrac{1}{2}B - \tan \tfrac{1}{2}A) = cr_1^2(\cot \tfrac{1}{2}B - \tan \tfrac{1}{2}C), \ldots\ldots \quad (5)$$

$$\frac{\sin \frac{1}{2}B \cos \frac{1}{2}B}{b} = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C}{c}, \ldots (6) \qquad \frac{\sin \frac{1}{2}A \cos \frac{1}{2}A}{a} = \frac{\sin \frac{1}{2}C \cos \frac{1}{2}C}{c} \ldots (7)$$

Dividing (1) by (2) and (3), and clearing of fractions, we have

$$b(\cot \tfrac{1}{2}A + 2r_1 + r_1^2 \cot \tfrac{1}{2}B) = c(\cot \tfrac{1}{2}A + 2r_2 + r_2^2 \cot \tfrac{1}{2}C), \ldots \quad (8)$$

$$a(\cot \tfrac{1}{2}A + 2r_1 + r_1^2 \cot \tfrac{1}{2}B) = c(r_1^2 \cot \tfrac{1}{2}B + 2r_1 r_2 + r_2^2 \cot \tfrac{1}{2}C). \quad (9)$$

Subtracting (4) from (8) and (5) from (9), we have

$$b(\tan \tfrac{1}{2}B + 2r_1 + r_1^2 \cot \tfrac{1}{2}B) = c(\tan \tfrac{1}{2}C + 2r_2 + r_2^2 \cot \tfrac{1}{2}C), \ldots \quad (10)$$

$$a(\cot \tfrac{1}{2}A + 2r_1 + r_1^2 \tan \tfrac{1}{2}A) = c(r_1^2 \tan \tfrac{1}{2}C + 2r_1 r_2 + r_2^2 \cot \tfrac{1}{2}C). \quad (11)$$

Multiplying (10) by (6) and (11) by (7), and extracting the square root, we have

$$\sin \tfrac{1}{2}B + r_1 \cos \tfrac{1}{2}B = \sin \tfrac{1}{2}C + r_2 \cos \tfrac{1}{2}C, \ldots\ldots \quad (12)$$

$$\cos \tfrac{1}{2}A + r_1 \sin \tfrac{1}{2}A = r_1 \sin \tfrac{1}{2}C + r_2 \cos \tfrac{1}{2}C. \ldots\ldots \quad (13)$$

Subtracting (13) from (12), we find

$$r_1 = \frac{\sin \frac{1}{2}C + \cos \frac{1}{2}A - \sin \frac{1}{2}B}{\sin \frac{1}{2}C - \sin \frac{1}{2}A + \cos \frac{1}{2}B} = \frac{\sin \frac{1}{2}C + \sin \frac{1}{4}C \cos \frac{1}{4}(2B + C)}{\sin \frac{1}{2}C + \sin \frac{1}{4}C \cos \frac{1}{4}(2A + C)}$$

$$= \frac{\cos \frac{1}{4}C + \cos \frac{1}{4}(2B + C)}{\cos \frac{1}{4}C + \cos \frac{1}{4}(2A + C)} = \frac{\cos \frac{1}{4}B \cos \frac{1}{4}(\pi - A)}{\cos \frac{1}{4}A \cos \frac{1}{4}(\pi - B)} = \frac{1 + \tan \frac{1}{4}A}{1 + \tan \frac{1}{4}B}.$$

Similarly we find

$$r_2 = \frac{\cos \frac{1}{4}C \cos \frac{1}{4}(\pi - A)}{\cos \frac{1}{4}A \cos \frac{1}{4}(\pi - C)} = \frac{1 + \tan \frac{1}{4}A}{1 + \tan \frac{1}{4}C}.$$

From (3) we have $x = \dfrac{a}{r_1{}^2 \cot \frac{1}{2}B + 2r_1 r_2 + r_2{}^2 \cot \frac{1}{2}C}$

$$= \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{r_1{}^2 \cos \frac{1}{2}B \sin \frac{1}{2}C + 2r_1 r_2 \sin \frac{1}{2}B \sin \frac{1}{2}C + r_2{}^2 \sin \frac{1}{2}B \cos \frac{1}{2}C}$$

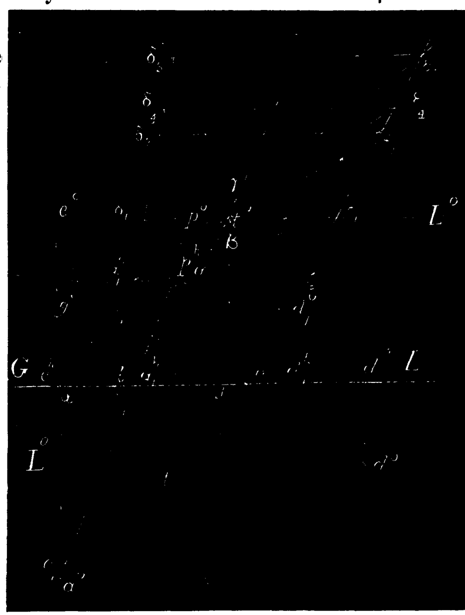
$$= \frac{r \cos \frac{1}{4}A \cos \frac{1}{4}(\pi - B) \cos \frac{1}{4}(\pi - C)}{2 \cos \frac{1}{4}\pi \cos \frac{1}{4}B \cos \frac{1}{4}C \cos \frac{1}{4}(\pi - A)} = \frac{\frac{1}{2}r(1 + \tan \frac{1}{4}B)(1 + \tan \frac{1}{4}C)}{1 + \tan \frac{1}{4}A}.$$

$$\therefore r_1{}^2 x = \frac{\frac{1}{2}r(1 + \tan \frac{1}{4}A)(1 + \tan \frac{1}{4}C)}{1 + \text{tsn} \frac{1}{4}B}, \; r_2{}^2 x = \frac{\frac{1}{2}r(1 + \tan \frac{1}{4}A)(1 + \tan \frac{1}{4}B)}{1 + \tan \frac{1}{4}C}.$$

SOLUTION OF MR. CHURCH'S PROBLEM.

BY PROF. E. W. HYDE, ITHACA, N. Y.

"Given four points [no one point lying within the triangle formed by the other three] to construct geometrically the axis and focus of the parabola passing through them.



Solution.—Let the four points be a^0, b, c, d^0. Regard them as lying in the horizontal plane of projection, and draw a ground line GL through two of them as b and c. Take some point p [marked $p^h p^v$] in the second angle, and draw the right lines pd and pa piercing the vertical plane in d_1 and a_1. b and c are in GL and therefore in both the horizontal and vertical planes of projection.

Draw in the vertical plane a horizontal line L^0 through p^v. Now if an ellipse be drawn passing through the four points $a^0{}_1$, b, c, and $d^0{}_1$, and also tangent to the line L^0, and if p be taken as the vertex of the projecting cone, the ellipse $a^0{}_1$ b c $d^0{}_1$ t^0 will be projected into the required parabola, upon the horizontal plane. For by the construction b and c are their own projections, $a^0{}_1$ is projected into a^0, and $d^0{}_1$ into d^0, and t^0 is projected to infinity. It follows therefore that p^h t^h is a diameter of the parabola. It is not necessary to construct the ellipse. Draw by Pascal's theorem a tangent to the ellipse at